Solution Sheet 12

Exercise 1. Let M be a smooth manifold of real dimension m, and let $(x^1, \dots, x^m)$ be local coordinates on an open set $U \subset M$. Any k-form on U can be written as

$$\omega = \sum_{1 \le i_1 < \cdots < i_k \le m} \omega_{i_1 \cdots i_k}\, dx^{i_1} \wedge \cdots \wedge dx^{i_k}, \qquad \omega_{i_1 \cdots i_k} \in C^\infty(U).$$

In these coordinates, the exterior derivative is given by

$$d\omega = \sum_{1 \le i_1 < \cdots < i_k \le m} \sum_{j=1}^{m} \frac{\partial \omega_{i_1 \cdots i_k}}{\partial x^j}\, dx^j \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_k}.$$

Use this formula to verify the defining properties of the exterior derivative $d : \Omega^k(U) \to \Omega^{k+1}(U)$:

(1) For $f \in \Omega^0(U) = C^\infty(U)$, the form df is the differential of f.
(2) For all $f \in \Omega^0(U)$ we have $d(df) = 0$.
(3) (Leibniz rule) For $\alpha \in \Omega^p(U)$ and $\beta \in \Omega^q(U)$ we have

$$d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^p\, \alpha \wedge d\beta.$$

(4) For every differential form $\omega \in \Omega^k(U)$ we have $d(d\omega) = 0$.

Solution 1.

(1) Applying the coordinate formula with $k = 0$ gives

$$df = \sum_{j=1}^{m} \frac{\partial f}{\partial x^j}\, dx^j,$$

which is the usual differential of f.

(2) Using the previous formula and antisymmetry of the wedge product we obtain

$$\begin{aligned} d(df) &= \sum_{a=1}^{m} \sum_{b=1}^{m} \frac{\partial^2 f}{\partial x^a\, \partial x^b}\, dx^a \wedge dx^b \\ &= \sum_{1 \le a < b \le m} \left(\frac{\partial^2 f}{\partial x^a\, \partial x^b} - \frac{\partial^2 f}{\partial x^b\, \partial x^a} \right) dx^a \wedge dx^b = 0. \end{aligned}$$

(3) For a multi-index $I = (i_1 < \cdots < i_p)$ we use the notation $dx^I = dx^{i_1} \wedge \cdots \wedge dx^{i_p}$. Consider $\alpha = f\, dx^I \in \Omega^p(U)$ and $\beta = g\, dx^J \in \Omega^q(U)$. Then

$$\alpha \wedge \beta = fg\, (dx^I \wedge dx^J).$$

Therefore

$$\begin{aligned} d(\alpha \wedge \beta) &= d(fg) \wedge dx^I \wedge dx^J \\ &= (g\, df + f\, dg) \wedge dx^I \wedge dx^J \\ &= g\, df \wedge dx^I \wedge dx^J + f\, dg \wedge dx^I \wedge dx^J \\ &= d\alpha \wedge \beta + (-1)^p\, \alpha \wedge d\beta, \end{aligned}$$

where in the last step we used

$$f\, dg \wedge dx^I \wedge dx^J = (-1)^p\, f\, dx^I \wedge dg \wedge dx^J = (-1)^p\, \alpha \wedge d\beta.$$

(4) Let $\omega = f\, dx^I$ be a k-form. Then $d\omega = df \wedge dx^I$ and the Leibniz rule gives

$$d(d\omega) = d(df \wedge dx^I) = d(df) \wedge dx^I - df \wedge d(dx^I).$$

By part (2), $d(df) = 0$. Moreover $d(dx^I) = 0$ as well. Therefore $d(d\omega) = 0$.

Exercise 2. Let $U \subset \mathbb{R}^2$ be a contractible open set. Show that every closed differential 1-form on U is exact.

Solution 2. Let $\omega = f(x,y)dx + g(x,y)dy$ be a closed 1-form. Then on U we have

$$\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} = 0.$$

Since U is contractible, we can assume that U is star-shaped with respect to some point $(x_0, y_0) \in U$. Define a function $\psi : U \to \mathbb{R}$ by

$$\psi(x,y) := \int_{x_0}^{x} f(t, y_0)dt + \int_{y_0}^{y} g(x,t)dt.$$

By the fundamental theorem of calculus we get

$$\frac{\partial \psi}{\partial y}(x,y) = g(x,y).$$

The partial derivative with respect to x gives

$$\frac{\partial \psi}{\partial x}(x,y) = f(x,y_0) + \int_{y_0}^{y} \frac{\partial g}{\partial x}(x,t)dt.$$

Using the closedness condition $\frac{\partial g}{\partial x} = \frac{\partial f}{\partial y}$, this becomes

$$f(x,y_0) + \int_{y_0}^{y} \frac{\partial f}{\partial y}(x,t)\,dt = f(x,y_0) + f(x,y) - f(x,y_0) = f(x,y).$$

It follows that

$$d\psi = \frac{\partial \psi}{\partial x}dx + \frac{\partial \psi}{\partial y}dy = fdx + gdy = \omega,$$

so ω is exact.

Exercise 3. (for credit, due on 21 December)
Let X be a Riemann surface and (U,z) a chart of X. Recall that the $*$ operator on 1-forms and the Laplace operator Δ taking 0-forms to 2-forms are defined by

$$*\omega = -i\omega^{(1,0)} + i\omega^{(0,1)},$$
$$\Delta f = -d * df.$$

We say that $\eta \in \Omega^1(U)$ is harmonic if it is locally of the form $\eta = df$ for $f \in \Omega^0(U)$ with $\Delta f = 0$.

(1) *(1 point)* Show that a 1-form $\eta \in \Omega^1(U)$ is harmonic if and only if $d\eta = 0$ and $d(*\eta) = 0$. **Hint:** Use the Poincaré lemma (Exercise 2).
(2) *(2 points)* Show that a 1-form $\eta \in \Omega^1(U)$ is harmonic if and only if it is of the form $\eta = \omega_1 + \overline{\omega_2}$ for some holomorphic 1-forms ω_1 and ω_2 on U.
(3) *(1 point)* Show that a 1-form $\omega \in \Omega^1(U)$ is holomorphic if and only if it is of the form $\omega = \eta + i * \eta$ for some harmonic 1-form η on U.
(4) *(1 point)* Show that every holomorphic 1-form on U is closed and harmonic.

Solution 3.

Exercise 4. Let X be a compact, connected Riemann surface of genus $g \geq 1$. Let P_g be its planar diagram (without glueing along edges). We denote by P_g° and ∂P_g the interior and boundary of P_g. Let $\{a_i, b_i,\ 1 \leq i \leq g\}$ be a canonical basis of cycles of $H_1(X, \mathbb{Z})$. For any closed 1-form α, we define its periods

$$A_i(\alpha) = \int_{a_i} \alpha, \quad B_i(\alpha) = \int_{b_i} \alpha$$

with respect to these cycles. The goal of the exercise is to prove Riemann's bilinear identity: If α, β are closed 1-forms, then

$$\int_X \alpha \wedge \beta = \sum_{i=1}^{g} \Big(A_i(\alpha) B_i(\beta) - B_i(\alpha) A_i(\beta) \Big).$$

(1) Explain why $\int_X \alpha \wedge \beta = \int_{P_g^\circ} \alpha \wedge \beta$.

(2) Show that there is a function $f \in C^\infty(P_g^\circ, \mathbb{C})$ such that $\int_X \alpha \wedge \beta = \int_{\partial P_g} f\beta$.

(3) Deduce Riemann's bilinear identity from the previous formula.

Solution 4.

(1) Since no two points of the interior of the planar diagram are identified by glueing, we can consider P_g° as an open subset of X. Since the complement $X \setminus P_g^\circ$ has measure zero in the surface, we obtain $\int_X \alpha \wedge \beta = \int_{P_g^\circ} \alpha \wedge \beta$.

(2) Fix $p_0 \in P_g^\circ$. Since P_g° is the interior of a $4g$-gon, it is simply connected. Define

$$f : P_g^\circ \to \mathbb{C}, \qquad f(p) := \int_\gamma \alpha,$$

where γ is any smooth path in P_g° from p_0 to p. This definition is well-defined: If γ_1, γ_2 are two such paths, then $\gamma_1 * \gamma_2^{-1}$ is a loop in P_g° and bounds a 2-chain $S \subset P_g^\circ$. By Stokes' theorem and $d\alpha = 0$ we get

$$\int_{\gamma_1} \alpha - \int_{\gamma_2} \alpha = \int_{\gamma_1 \cdot \gamma_2^{-1}} \alpha = \int_{\partial S} \alpha = \int_S d\alpha = 0,$$

so f is independent of the choice of γ. Furthermore we have $df = \alpha$ on P_g°. Namely, for any smooth curve $c(t)$ in P_g° with $c(0) = p$, concatenate a fixed path from p_0 to p with $c|_{[0,t]}$ to obtain a path from p_0 to $c(t)$. This gives

$$f(c(t)) - f(p) = \int_{c|_{[0,t]}} \alpha,$$

and then differentiating at $t = 0$ yields

$$\frac{d}{dt}\Big|_{t=0} f(c(t)) = \frac{d}{dt}\Big|_{t=0} \int_{c|_{[0,t]}} \alpha = \alpha_p(c'(0)).$$

Since this holds for every curve c with $c(0) = p$, we conclude that $(df)_p = \alpha_p$ for all p, hence $df = \alpha$. Since $d\beta = 0$, we have

$$d(f\beta) = df \wedge \beta + f\, d\beta = df \wedge \beta = \alpha \wedge \beta.$$

Therefore by Stokes' theorem we get

$$\int_{P_g^\circ} \alpha \wedge \beta = \int_{P_g^\circ} d(f\beta) = \int_{\partial P_g} f\beta.$$

(3) From $\partial P_g = \prod_{i=1}^g a_i b_i a_i^{-1} b_i^{-1}$ we have the decomposition

$$\int_{\partial P_g} f\beta = \sum_{i=1}^{g} \left(\int_{a_i} f\beta + \int_{a_i^{-1}} f\beta + \int_{b_i} f\beta + \int_{b_i^{-1}} f\beta \right).$$

Let $\tau_{a_i} : a_i \to a_i^{-1}$ be the gluing map. It reverses orientation and β agrees on the two identified edges. Hence

$$\int_{a_i^{-1}} f\beta = -\int_{a_i} (f \circ \tau_{a_i})\beta,$$

and therefore

$$\int_{a_i} f\beta + \int_{a_i^{-1}} f\beta = \int_{a_i} \big(f - f \circ \tau_{a_i}\big)\beta.$$

For $p \in a_i$, set $q = \tau_{a_i}(p)$. Since $df = \alpha$ on P_g° for any path $\gamma \subset P_g^\circ$ from p to q we have $f(q) - f(p) = \int_\gamma \alpha$, and the corresponding loop in X represents b_i. Therefore $f \circ \tau_{a_i} - f = B_i(\alpha)$ on a_i, and hence

$$\int_{a_i} f\beta + \int_{a_i^{-1}} f\beta = -B_i(\alpha) \int_{a_i} \beta = -B_i(\alpha)A_i(\beta).$$

Similarly we get

$$\int_{b_i} f\beta + \int_{b_i^{-1}} f\beta = A_i(\alpha) \int_{b_i} \beta = A_i(\alpha)B_i(\beta).$$

Summing over i yields

$$\int_{\partial P_g} f\beta = \sum_{i=1}^{g} \big(A_i(\alpha)B_i(\beta) - B_i(\alpha)A_i(\beta)\big),$$

and we conclude with $\int_X \alpha \wedge \beta = \int_{\partial P_g} f\beta$.